As shown in class, we can use induction to prove sums if given sequences as well as to determine whether or not the *n*th term is divisible by some integer, however there are many more applications of induction. Here I will give you some more examples of using mathematical induction so you can further understand the mechanics behind it and understand when to use it. (You **will not** be expected to be able to do all the examples **nor anything in the *Additional Information* sections**– they are just for you to further develop your understanding of mathematical induction.)

Example 1 *Prove by induction that* $\underline{5^{2n} - 1 \text{ is a multiple of } 8}$ *for all* $n \in \mathbb{N}$.

Solution:

1) Begin by showing the underlined statement is true for some value of n; we can use $n = 1$.
$$5^{2(1)} - 1 = 5^2 - 1 = 25 - 1 = 24 = 8(3)$$
which is clearly a multiple of 8, so the underlined statement is true for $n = 1$.

2) Now if we assume that $5^{2k} - 1$ is a multiple of 8 $(*)$, we show that $5^{2(k+1)} - 1$ is a multiple of 8 $(**)$.
Then for some number *x*,
$$\begin{aligned}
5^{2k} - 1 &= 8x && \text{(given as true)}\\
5^{2k} &= 1 + 8x && \text{(add 1 to both sides)}\\
5^2\left(5^{2k}\right) &= 5^2(1 + 8x) && \text{(multiply both sides by } 5^2\text{)}\\
5^{2k+2} &= 25 + 25(8x) && \text{(combine exponents and distribute)}\\
5^{2(k+1)} &= 1 + 24 + 8(25x) && \text{(TRICKY STEP – rewrite 25 as 24+1)}\\
5^{2(k+1)} - 1 &= 24 + 8(25x) && \text{(subtract one from both sides)}\\
5^{2(k+1)} - 1 &= 8(3 + 25x) && \text{(factor out 8 on RHS)}
\end{aligned}$$

We see that the right hand side (RHS) of the equation can be divided by 8 which means it is a multiple of 8. Therefore, since both sides of the equation are equal, the left hand side (LHS) is also a multiple of 8, which is what we wanted to show. From parts 1) and 2), we may conclude that the statement is true for all $n \in \mathbb{N}$.

***ADDITIONAL INFORMATION**: Alternate proof for part 2):* we can start with $(**)$ and show $(*)$.
Then for some integer *x*,
$$\begin{aligned}
5^{2(k+1)} - 1 &= 8x && \text{(claim to be true and we are proving true)}\\
5^2 5^{2k} &= 1 + 8x && \text{(add 1 to both sides and use addition of powers property of exponents)}\\
5^{2k} &= \tfrac{1+8x}{25} && \text{(divide both sides by 25)}\\
5^{2k} - 1 &= \tfrac{1+8x}{25} - 1 && \text{(TRICKY STEP – subtract 1 from both sides)}\\
5^{2k} - 1 &= \tfrac{1+8x-25}{25} && \text{(find common denominator and write as one fraction)}\\
5^{2k} - 1 &= \tfrac{8x-24}{25} = \tfrac{8(x-3)}{25} && \text{(rewrite and factor out 8 on RHS)}
\end{aligned}$$
We know that the LHS is a multiple of 8 by our assumption (at the beginning of part 2) and the RHS is also a multiple of 8, so we can also conclude that the statement is true for all $n \in \mathbb{N}$.

Example 2 *Prove by induction that*
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \left(\text{or equivalently, } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right)$$
for all $n \in \mathbb{N}$.

Solution:

1) Again we show that the statement is true for some value of n; for simplicity, show for $n = 1$.
$$1^2 = \frac{1(1+1)(2*1+1)}{6}$$

$$1 = \frac{1(2)(3)}{6}$$
$$1 = 1$$

Which is always true.

2) Now we assume that the given statement is true for n and show that it is true for $(n+1)$, meaning we want to show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

**Note: we are using *n* here instead of *k* – the process is exactly the same as before; just with a different variable!

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(given to be true)}$$
$$(1^2 + 2^2 + 3^2 + \cdots + n^2) + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \text{(add } (n+1)^2 \text{ to both sides)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)+6(n+1)^2}{6} \quad \text{(common denominator)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)[n(2n+1)+6(n+1)]}{6} \quad \text{(factor out } (n+1))$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)[2n^2+7n+6]}{6} \quad \text{(expand)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} \quad \text{(factor trinomial)}$$

Which is what we wanted to show! Therefore, from 1) and 2), we can say that the sequence is true for all $n \in \mathbb{N}$ by induction.

ADDITIONAL INFORMATION: *Alternate proof of part 2)*: Assume the given statement is true for *n* and show that it is true for *n+1*:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6} \quad \text{(what we want to show)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)(n+2)(2n+3)}{6} - (n+1)^2 \quad \text{(subtract } (n+1)^2 \text{ from both sides)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)(n+2)(2n+3)-6(n+1)^2}{6} \quad \text{(common denominator)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)[(n+2)(2n+3)-6(n+1)]}{6} \quad \text{(factor out } (n+1))$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)[2n^2+n]}{6} \quad \text{(expand and collect terms)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)n(2n+1)}{6} \quad \text{(factor out n)}$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{(rewrite)}$$

Which is true by our assumption. Therefore, the sequence is true for all $n \in \mathbb{N}$ by induction.

<u>Induction with Inequalities</u>

These are very tricky proofs and until you are extremely proficient in using mathematical induction, they remain to be some of the hardest proofs! First, try to get the side that is greater to the desired form. After this, use inequalities to decrease the other side until it becomes what you need it to be. Alternatively, you can get the "less than side" to what you want it to be and use inequalities to increase the other side until you get what you need it to be. MAKE SURE WHAT YOU ARE DOING IS VALID WITH THE GIVEN INFORMATION!!

***Example 3** *Prove by induction that* $\underline{2^n \geq 7n}$ *for all* $n \geq 6$

Solution:

1) Notice the important restriction! So instead of showing *n=1* true, we start with showing *n=6* true!

$$2^6 \geq 7(6)$$
$$64 \geq 42$$

Which is always true.

2) Now we assume the statement is true for k $(6 \leq k \leq n)$, and we show the statement is true for *(k+1)*, meaning that we want to show:

$$2^{k+1} \geq 7(k+1)$$

So we start with what we know to be true:

$$2^k \geq 7k$$
$$2^k + 7 \geq 7k + 7 \quad \text{(add 7 to both sides)}$$
$$2^k + 7 \geq 7(k+1) \quad \text{(factor RHS)}$$

Now we could be stuck…but let's try using inequalities to increase the LHS:

$$2 * 2^k \geq 2^k + 7$$
$$2^{k+1} \geq 2^k + 7$$

This step is not completely obvious immediately, but we use logic to reason that multiplication is stronger than addition for large values of k to make this claim! (Again, this is tricky!)
So now we rewrite everything:

$$2^{k+1} \geq 2^k + 7 \geq 7(k+1)$$
$$2^{k+1} \geq 7(k+1)$$

Which is what we wanted to show! Therefore we conclude that the underlined statement is true because of 1) and 2) for all $n \geq 6$.

ADDITIONAL INFORMATION: Recursive sequences

In addition to having sequences defined by certain mathematical operations, sequences can be defined in terms of previous terms in the sequence! These types of sequences are called **recursive sequences**. In these sequences, you are given an initial value for the sequence and with that value, you can generate the entire sequence with the given formula. (A great example of a recursive sequence is the Fibonacci sequence – look it up or ask if you are unfamiliar with it.) These types of sequences also utilize mathematical induction to prove them!

***Example 4 (Tower of Hanoi – moving blocks to peg 2)[1]** *Given that the recursive sequence is defined by $a_n = 2a_{n-1} + 1$ with initial value $a_1 = 1$, prove by induction that the value of the* nth *term in the a_n sequence can be determined by the equation $b_n = 2^n - 1$.*

Solution:
First let's write out a few terms of the sequence so we have an idea of what the sequence looks like (using the recursive sequence definition, a_n):

$$1, 2(1) + 1 = 3, 2(3) + 1 = 7, 2(7) + 1 = 15, \ldots$$
$$1, 3, 7, 15, \ldots$$

Okay, now begin the induction process on b_n.

1) Show that the statement is true for some value of n; to change things up, show for $n = 2$. (We can do $n = 1$ as well – we can attempt to show the *b* sequence true for any *n*, but we must know that value!)

$$b_2 = 2^2 - 1 = 4 - 1 = 3 = a_2$$

Which is true, so now we move onto the second step…

2) Now we assume that the underlined statement is true for k $(1 \leq k \leq n)$ and we show it is true for *(k+1)*. So, we want to show that $b_{k+1} = 2^{k+1} - 1$ is true.

$$b_k = 2^k - 1 \quad \text{(given)}$$
$$2b_k = 2 * 2^k - 2(1) \quad \text{(multiply both sides by 2)}$$
$$2b_k = 2^{k+1} - 2 \quad \text{(addition of powers and distribution)}$$
$$2b_k = 2^{k+1} - 1 - 1 \quad \text{(TRICKY STEP \#1 – rewrite -2 as -1-1)}$$
$$2b_k + 1 = 2^{k+1} - 1 \quad \text{(add 1 to both sides)}$$
$$b_{k+1} = 2^{k+1} - 1 \quad \text{(TRICKY STEP \#2 – use the definition of the recursive sequence } a_n \text{ on LHS)}$$

Which was our goal to show! Therefore by 1) and 2) the underlined statement is true for all *n*.

[1] (White, 2008)

ADDITIONAL INFORMATION: *Alternate proof of part 2):*

$b_{k+1} = 2^{k+1} - 1$ (claim to be true)

$b_{k+1} = 2 * 2^k - 1$ (property of exponents)

$2b_k + 1 = 2 * 2^k - 1$ (TRICKY STEP – use the definition of the recursive sequence a_n on LHS)

$2b_k = 2 * 2^k - 2$ (subtract 1 from both sides)

$b_k = 2^k - 1$ (divide everything by 2)

Which we assume to be true, therefore the underlined statement is true for all *n*.

**Remark: Look at the steps of the alternate proof in Example 3 in reverse and compare to the part 2 proof. What do you notice?